Stat 274
Theory of Interest

Lecture 1: The Growth of Money

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At 5% (annual compound) interest how much does 1,000 grow to in

- one year?

$1000(1+0.05) = 1000(1.05)$

$1000 + 1000(0.05) = 1050$

- two years?

$1050(1.05) = 1000(1.05)^2$

- t years?

$1000(1.05)^t$

Interest (no math allowed)

500 in 20 years at 5% interest will grow to 1,326.65. How much would:

$500(1.05)^{20}$

$250(1.05)^{20}$

- 250 become?

  663.32

- 1,000 become?

  2653.30

Interest (no math allowed)

500 in 20 years at 5% interest will grow to 1,326.65. How much would it grow to in:

- 10 years?

  814.45

- 40 years?

  3519.99

Interest (no math allowed)

500 in 20 years at 5% interest will grow to 1,326.65. How much would it grow to at:

- 2.5% interest?

  819.31

- 10% interest?

  3363.75

What is interest?

An investment of K grows to S, then the difference $(S - K)$ is the interest.

Why do we charge interest?

- Investment opportunities theory
- Time preference theory
- Risk premium

Should we charge interest?

Basic Definitions

Principal, K: The amount of money loaned by the investor, unless otherwise specified it is loaned at time $t = 0$.

$500(1.05)^{20} = 1326 = A_{500}(20)$

Amount function, $A_K(t)$: the value of K principal at time t.

$1(1.05)^{20} = a(20)$

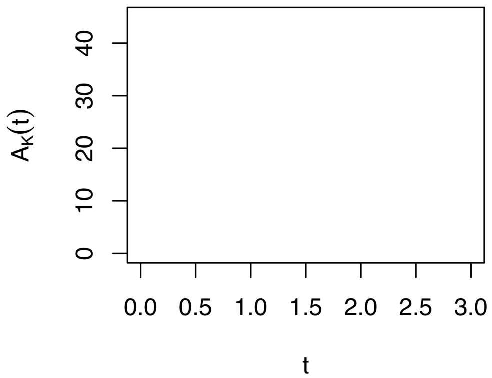
Accumulation function, $a(t)$: the value of 1 at time t,
$a(t) = A_1(t)$.

Often, $A_K(t) = Ka(t)$.

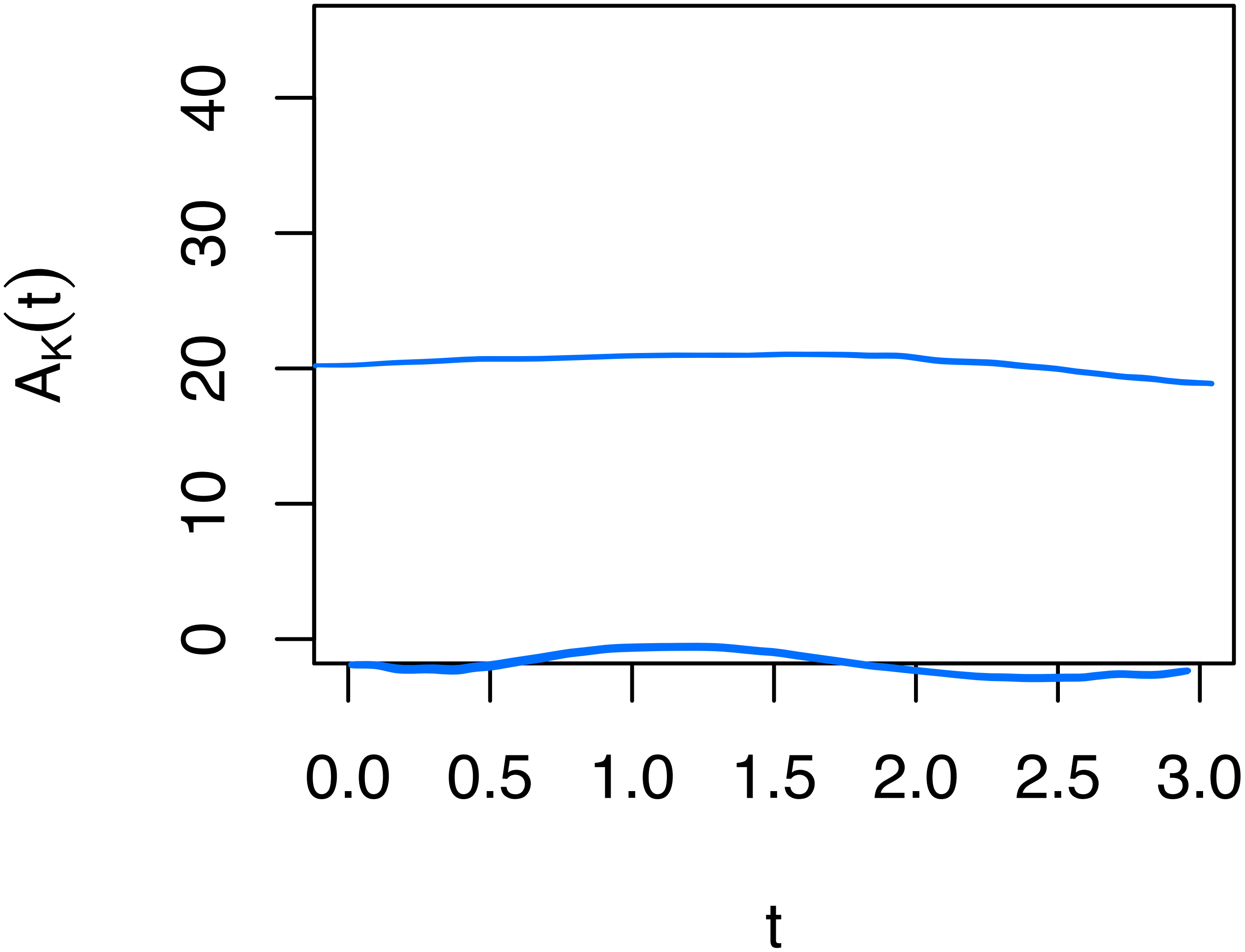
- What does that mean?
- When is this not true?

Examples

1. Suppose you borrow 20 from your parents, what would the amount owed look like over time?
2. Suppose you borrow 20 from your friend, what would the amount owed look like over time?
3. Suppose you borrow 20 from your bank, what would the amount owed look like over time?
4. Suppose you borrow 20 from a loan shark, what would the amount owed look like over time?
5. Suppose you deposit 20 into a bank which earns 1 at the end of every year (but nothing during the year), what would the account balance look like over time?

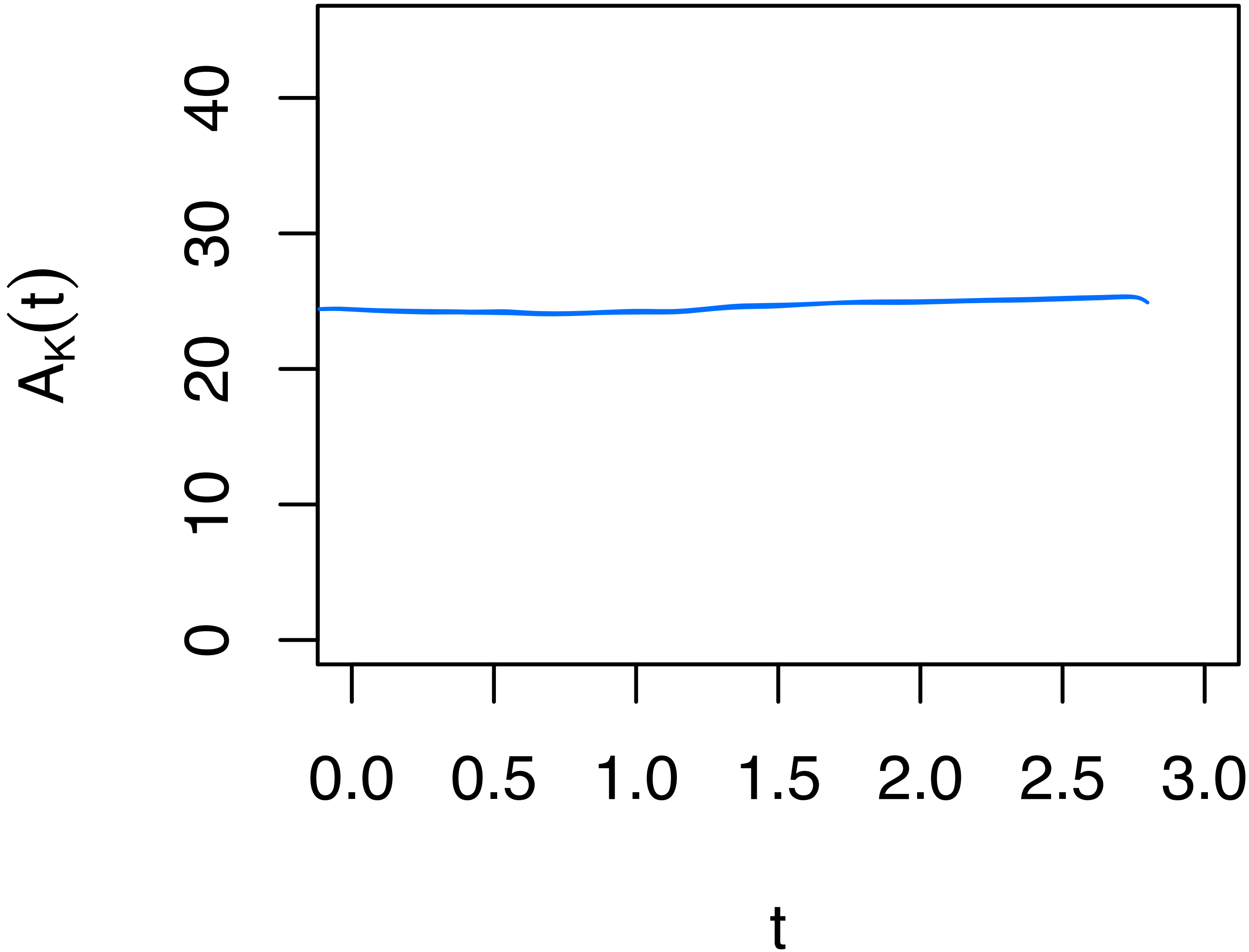


Suppose you borrow 20 from your parents, what would $A_K(t)$ look like?



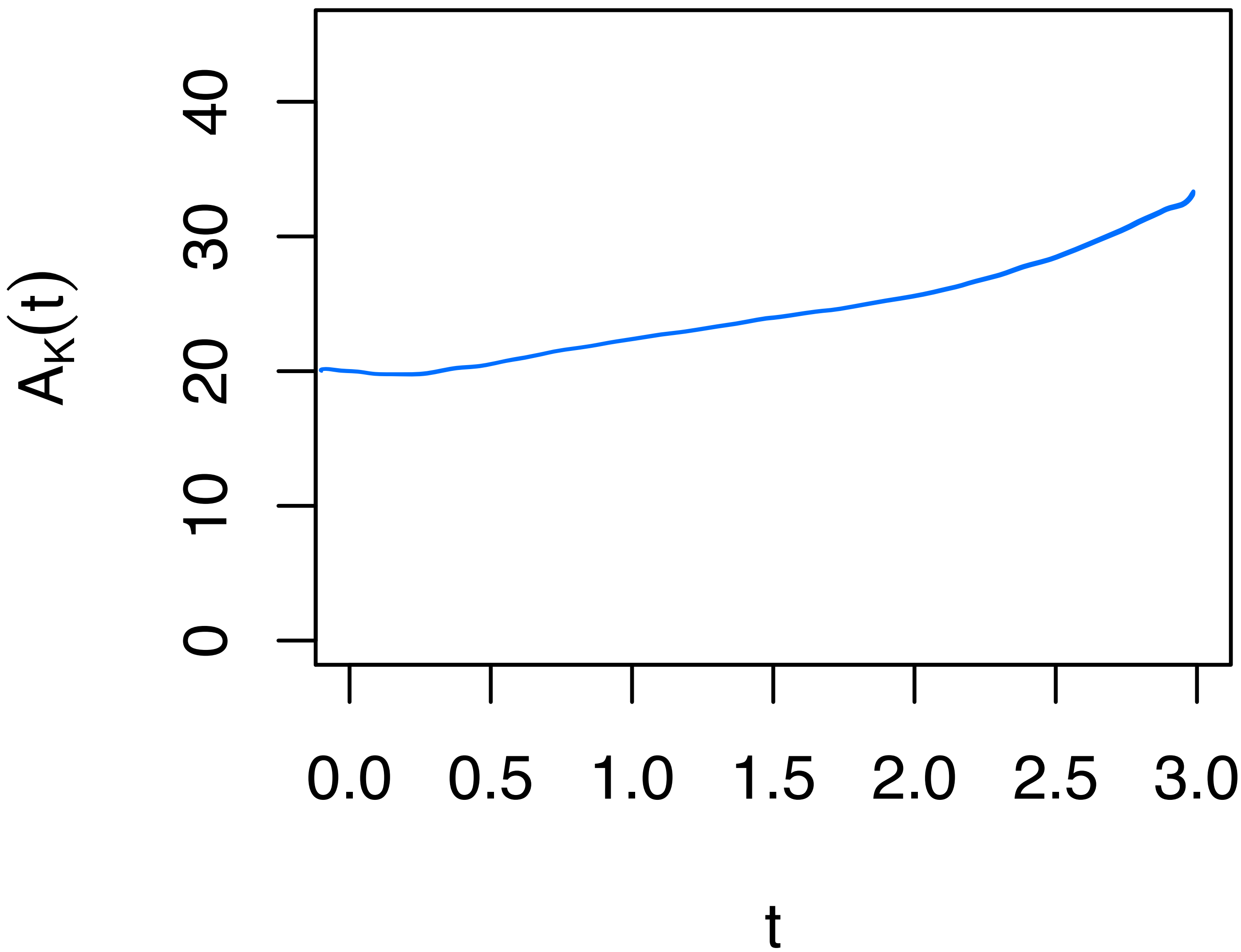
Examples

Suppose you borrow 20 from your friend, what would $A_K(t)$ look like?



Examples

Suppose you borrow 20 from your bank, what would $A_K(t)$ look like?



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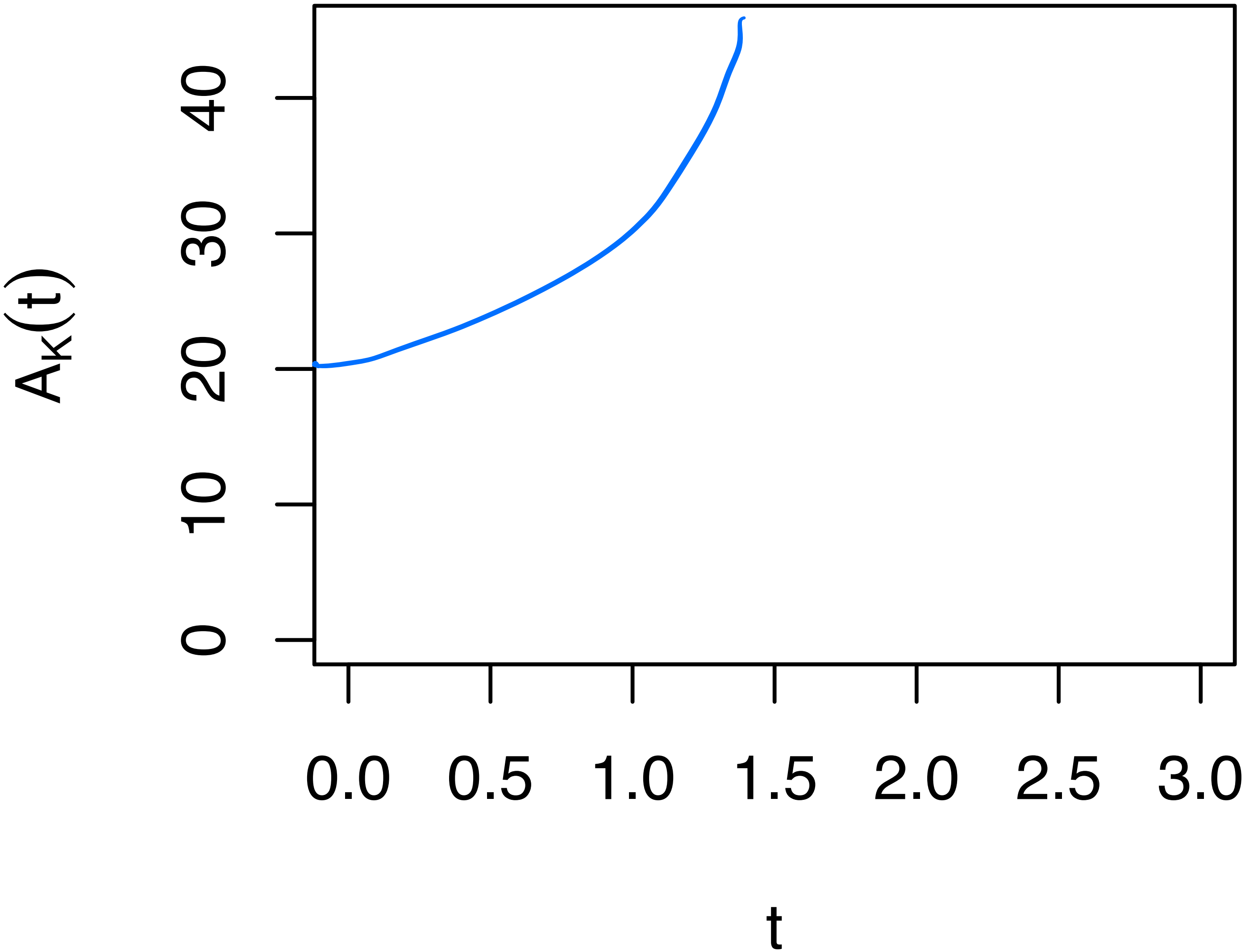

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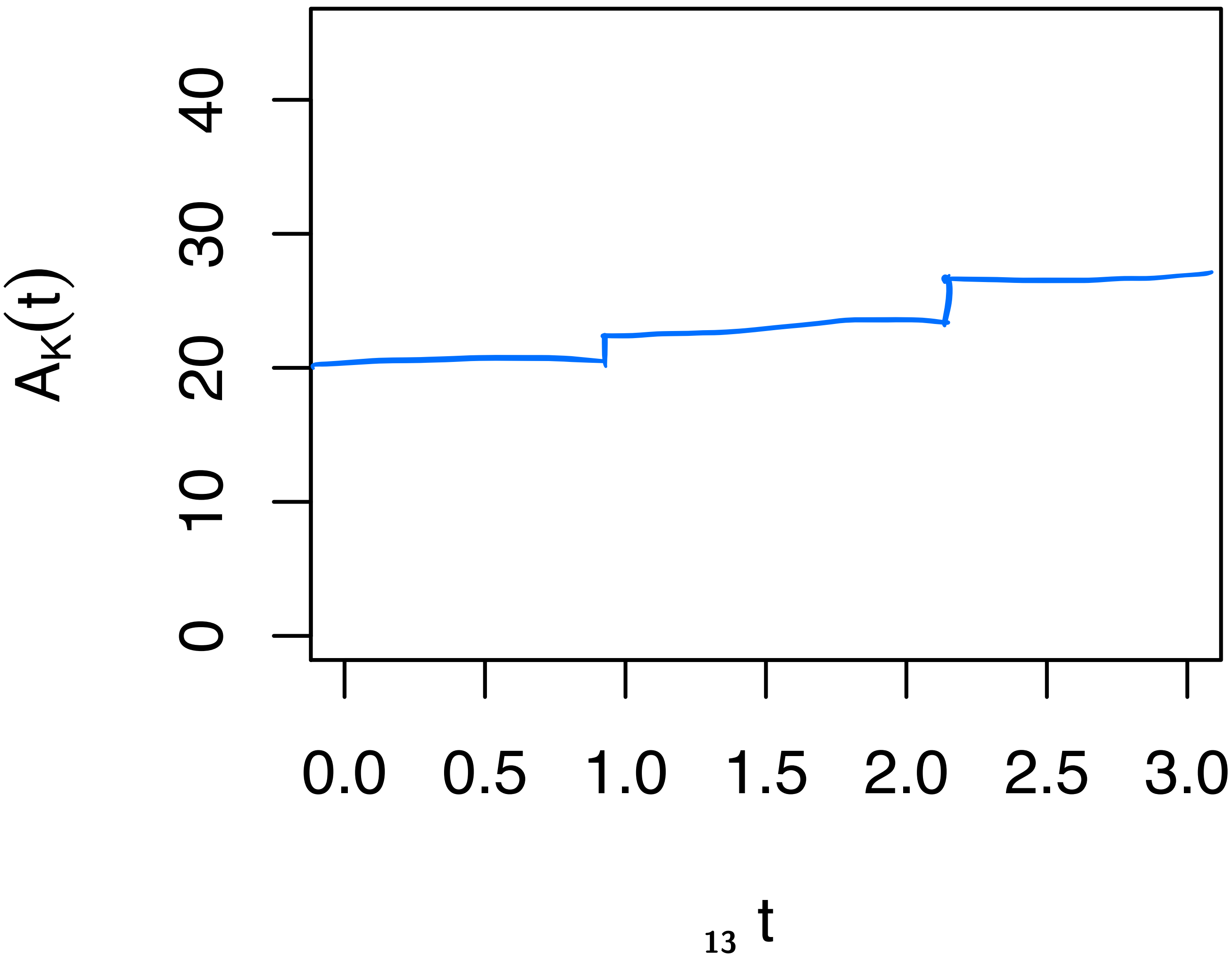


Effective Interest in Intervals

When $0 \leq t_1 \leq t_2$, the effective interest rate for $[t_1, t_2]$ is

$$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}$$

and if $A_K(t) = Ka(t)$ then

$$i_{[t_1,t_2]} = \frac{A_K(t_2) - A_K(t_1)}{A_K(t_1)}$$

Effective Interest in Intervals, Alternatively

Alternatively, when n is an integer, we can write i_n for $i_{[n-1,n]}$ leading to

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)}$$

and

$$a(n) = a(n-1)(1 + i_n)$$

How would this simplify for i_1?

Compound Interest

Most contracts use compound interest.

- Amount function: $A_K(t) = K(1+i)^t$
- Accumulation function: $a(t) = (1+i)^t$
- Effective interest rate: $i_n = i$

Simple Interest

When an investment grows linearly over time, it is called simple interest.

- Amount function: $A_K(t) = K(1 + it)$ Ki
- Accumulation function: $a(t) = 1 + it$
- Effective interest rate: $i_n = \frac{i}{1+i(n-1)}$

Effective interest rates for simple interest

2400 is loaned at 5% simple interest for three years. The annual effective rates are:

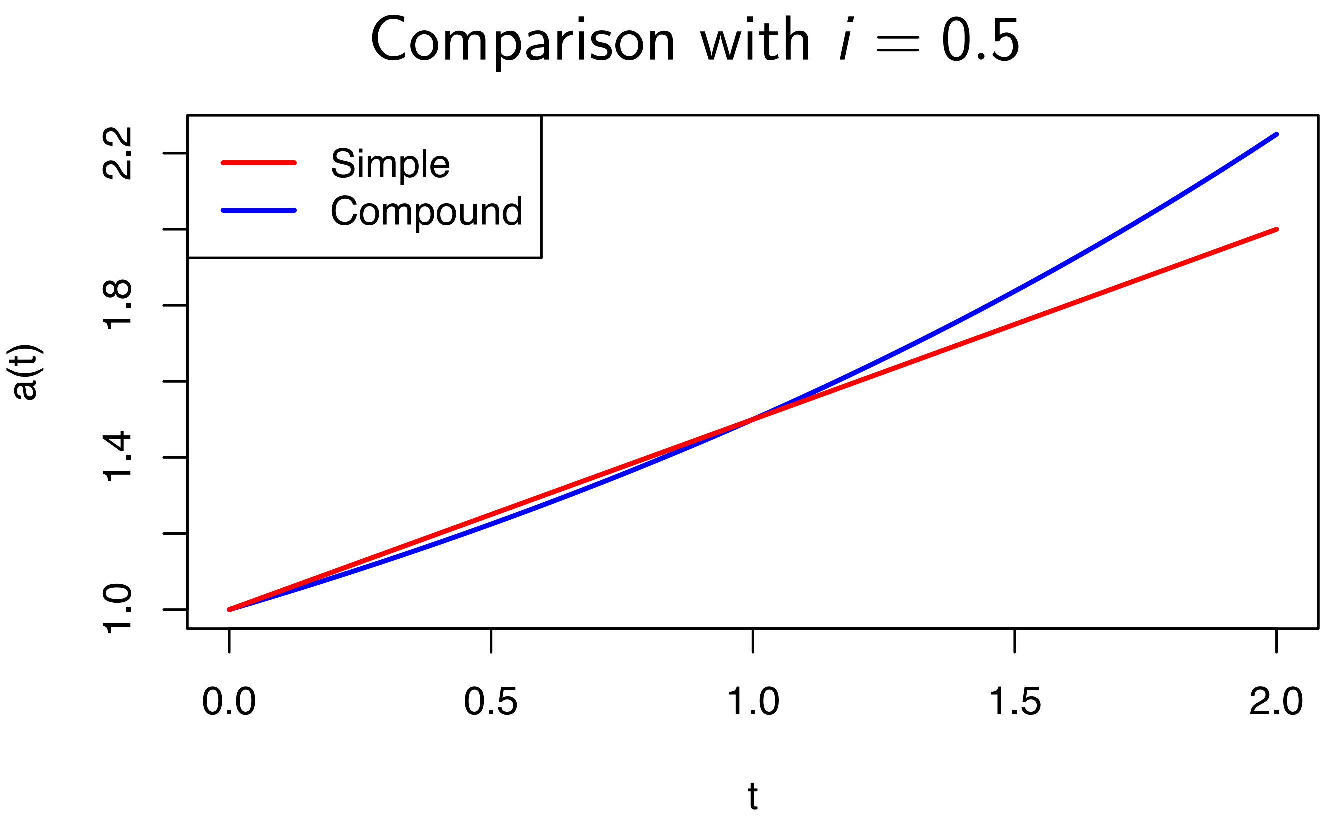
$$i_1 = \frac{2520 - 2400}{2400} = 5\%$$

$$i_2 = \frac{2640 - 2520}{2520} \approx 4.76\%$$

$$i_3 = \frac{2760 - 2640}{2640} \approx 4.55\%$$

How could you improve those rates?

Simple vs. Compound Interest



Examples

1. Given $A_K(t) = \frac{1000}{50-t}$ for $0 \leq t < 50$, calculate K and $a(10)$, assuming that $A_K(t) = Ka(t)$. [20, 25/20]

$$A_k(0) = K = \frac{1000}{50} = 20$$

$$A_k(10) = Ka(10)$$

$$\frac{A_k(10)}{K} = a(10)$$

2. For a loan of 1000, 1300 is repaid in three years. The money was loaned at what rate of simple interest? [10%]

$$1000(1+3i) = 1300$$

$$i = 0.10$$

Compound Interest Examples

An account is opened with 12000 and is closed in 6.5 years. The account earns 5% interest. How much is withdrawn from the account if

- Compound interest is paid throughout. [16478.27]

$$12000(1.05)^{6.5}$$

- Compound interest is paid on each whole year and then simple interest is paid on the last half year. [16483.18]

$$12000(1.05)^{6}\left(1+(0.5)(0.05)\right)$$

Assume an account pays 2% compound interest on balances less than 2000, 3% compound interest on balances between 2000 and 5000, and 4% compound interest on balances above 5000. What is $A_{1800}(t)$?

$1800(1.02)^t = 2000$

$1.02^t = \frac{2000}{1800}$ $\quad t \log(1.02) = \log\left(\frac{2000}{1800}\right)$

$t = \frac{\log\left(\frac{2000}{1800}\right)}{\log(1.02)}$

$$A_{1800}(t) = \begin{cases} 1800(1.02)^t & t < 5.32 \\ 2000(1.03)^{t-5.32} & 5.32 \leq t < 36.32 \\ 5000(1.04)^{t-36.32} & 36.32 \leq t \end{cases}$$

Examples

1. Assume that 1000 is deposited into an account. The effective annual compound interest rate is 3% for the first year, 4% for the next two, and 1% for the next three. How much would be in the account at the end of the six years? [1147.80]
2. Suppose you want to have 1000 in three years. You currently have 900 to invest. What interest rate (annually compounding) do you need to accomplish your goal? [3.57%]
3. Suppose you want to have 1000 in three years. If you could earn 2% annually compounding interest, how much would you need to invest to accomplish your goal? [942.32]

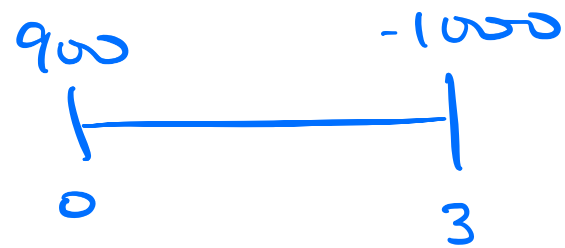
Assume that 1000 is deposited into an account. The effective annual compound interest rate is 3% for the first year, 4% for the next two, and 1% for the next three. How much would be in the account at the end of the six years? [1147.80]

$$1000(1.03)(1.04)(1.04)(1.01)(1.01)(1.01)$$

$$1000(1.03)(1.04)^2(1.01)^3$$

Examples

Suppose you want to have 1000 in three years. You currently have 900 to invest. What interest rate (annually compounding) do you need to accomplish your goal? [3.57%]



$$900(1+i)^3 = 1000$$

$$i = \left(\frac{1000}{900}\right)^{1/3} - 1$$

Suppose you want to have 1000 in three years. If you could earn 2% annually compounding interest, how much would you need to invest to accomplish your goal? [942.32]

$$1000 = x(1.02)^3$$

$$\frac{1000}{1.02^3} \quad x$$

Discount Rates

$$i = \frac{a(t_2) - a(t_1)}{a(t_1)}$$

Discount rates use the end of period accumulation, rather than the beginning of period.

$$d_{[t_1, t_2]} = \frac{a(t_2) - a(t_1)}{a(t_2)}$$

$i = 0.10$

$d = {}^{10}/_{110} = 0.0909$

If $A_K(t) = Ka(t)$ then

$$d_{[t_1,t_2]} = \frac{A_K(t_2) - A_K(t_1)}{A_K(t_2)}$$

Similar to i_n, when n is a positive integer,

$$d_n = \frac{a(n) - a(n-1)}{a(n)} \qquad \text{and} \qquad a(n-1) = a(n)(1 - d_n)$$

$a(n-1)(1+i) = a(n)$

Equivalence of Interest and Discount Rates

$i = \frac{d}{1-d}$ $\qquad$ $d = \frac{i}{1+i}$

Two rates are equivalent if they correspond to the same accumulation function.

$$1 = \left(1 + i_{[t_1,t_2]}\right)\left(1 - d_{[t_1,t_2]}\right)$$

$$i_{[t_1,t_2]} = \frac{d_{[t_1,t_2]}}{1 - d_{[t_1,t_2]}} \qquad \text{and} \qquad d_{[t_1,t_2]} = \frac{i_{[t_1,t_2]}}{1 + i_{[t_1,t_2]}}$$

Similarly,

$$i_n = \frac{d_n}{1 - d_n} \qquad \text{and} \qquad d_n = \frac{i_n}{1 + i_n}$$

$a(n-1)(1+i) = a(n)$

$a(n-1) = a(n)(1-d)$

$a(n)(1-d)(1+i) = a(n)$

$(1-d)(1+i) = 1$

Time Value of Money

100 now is worth more than 100 in three years. The value today of 100 in three years is determined by the discount function

$$v(t) = \frac{1}{a(t)}$$

When using the compound interest accumulation function, $a(t) = (1+i)^t$, we can define the discount factor

$$v = \frac{1}{1+i}$$

$$900(1+i)^3 = 1000$$

and show that

$$v(t) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = v^t$$

$$900 = \frac{1000}{(1+i)^3}$$

$$900 = 1000v^3$$

Compound Discount

Now, if d is constant we have

$$i = \frac{d}{1-d}$$

and

$$d = \frac{i}{1+i} = iv$$

1. You need 3000 today to pay tuition. You can borrow money at a 4% annual discount rate and will repay the money when you graduate in three years. How much will you repay when you graduate? [3390.84]

$i = \frac{d}{1-d}$

3000 $-x$

0   3

$3000(1+i)^3 = x$

$x(1-d)^3 = 3000$

$x = 3000(1-d)^{-3}$

2. You are going to receive a bonus of 100 in five years. You would like to sell that bonus today at a discount rate of no more than 5%. What is the smallest amount you would accept today? [77.38]

x -100

0   5

	i	d
Forward	$(1+i)^t$	$(1-d)^{-t}$
Backward	$\frac{1}{(1+i)^t} = v^t$	$(1-d)^t$

$100(1-d)^5 = x$

$100v^5 = x$

$100 = x(1+i)^5$

$100 = x(1-d)^{-5}$

$i = i^{(1)}$

$100(1+i)^{1.5} = 100\left(1+\frac{i^{(2)}}{2}\right)^3$
$= 100\left(1+\frac{i^{(12)}}{12}\right)^{18}$

Often, interest is credited more often than annually. The monthly (or quarterly, semi-annually, etc.) nominal interest rate is denoted $i^{(m)}$ where the m is the number of payments per year.

$i = 0.08 \qquad i^{(12)} = 0.077$

The nominal rates are per year, so you earn $\frac{i^{(m)}}{m}$ in interest every period. To find the equivalent nominal interest rate, we use the following fact:

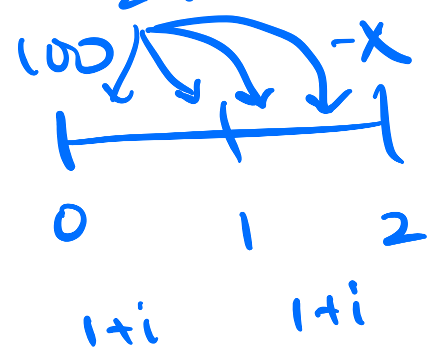
$\cancel{\left(1+ i^{(m)}\right)}$

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

$i^{(m)}$ - nominal

$\frac{i^{(m)}}{m}$ - effective

$\left(1+\frac{i^{(2)}}{2}\right)^4$



$\frac{i^{(12)}}{12}$ - monthly effect. int. rate

$\frac{i^{(2)}}{2}$

150000

0 —————————————— 3

$i = 0.12$

$i^{(4)} = 0.07$

$i^{(12)} = 0.06$

$$\left(1 + \frac{i^{(m)}}{m}\right)^{m \cdot 3}$$

$150000(1.12)^3$ [210,739.20]

$150000\left(1 + \frac{0.07}{4}\right)^{4 \cdot 3}$ [184,715.90]

$150000\left(1 + \frac{0.06}{12}\right)^{36}$ [179,502.08]

Similar facts exist for nominal discount rates, most importantly

$$(1-d)^{-1} = \left(1 - \frac{d^{(m)}}{m}\right)^{-m}$$

and

$$d^{(m)} = m\left[1 - (1-d)^{1/m}\right]$$

Equating Nominal Discount and Interest

$$\left(1-\frac{d^{(m)}}{m}\right)^{-m} = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$

We can derive the following few relationships

$$\left(1-\frac{d^{(m)}}{m}\right)\left(1+\frac{i^{(m)}}{m}\right) = 1$$

$$i^{(m)} = \frac{d^{(m)}}{1-\frac{d^{(m)}}{m}} \quad \text{and} \quad d^{(m)} = \frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}$$

and most generally

$$\left(1+\frac{i^{(n)}}{n}\right)^{n} = 1+i = (1-d)^{-1} = \left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

Nominal Rate Examples

If I invest 100 today and it grows to 115 in one year, what is the

1. annual simple interest rate? [0.15] i

   $100(1+it)=115 \qquad 100(1+i)=115 \qquad i=0.15$

2. annual compound interest rate? [0.15] i

   $100(1+i)^t=115 \qquad 100(1+i)=115 \qquad i=0.15$

3. nominal interest compounded monthly? [0.1406] $i^{(12)}$

   $100\left(1+\frac{i^{(12)}}{12}\right)^{12t}=115 \qquad \left[\left(\frac{115}{100}\right)^{1/12}-1\right]12=i^{(12)}$

4. nominal discount compounded monthly? [0.1389] $d^{(12)}$

   $100\left(1-\frac{d^{(12)}}{12}\right)^{-12t}=115 \qquad \left[1-\left(\frac{115}{100}\right)^{-1/12}\right]\cdot 12$

5. annual compound discount rate? [0.1304] d

   $100(1-d)^{-1t}=115 \qquad 1-\left(\frac{115}{100}\right)^{-1}=d$

$(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m \qquad \left[(1+i)^{1/m} - 1\right]m = i^{(m)}$

What happens as m increases?

$\lim_{m\to\infty} \frac{(1+i)^{1/m} - 1}{1/m}$

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} m\left[(1+i)^{1/m} - 1\right] = log(1+i) = \delta$$

$e^{\log((1+i)^{1/m})}$

$e^{\frac{1}{m}\log(1+i)}$

Further,

$$i = e^{\delta} - 1 \quad \text{and} \quad e^{\delta} = 1 + i$$

Which results in an accumulation function of $\log(1+i)\frac{-1}{m^2}\left(e^{\frac{1}{m}\log(1+i)}\right)$

$$a(t) = e^{\delta t}$$

Note that if $i > 0$ and $m > 1$ then $\lim_{m\to\infty} \frac{\log(1+i)\cancel{\left(\frac{-1}{m^2}\right)}e^{\frac{1}{m}\log(1+i)}}{\cancel{-1/m^2}}$

$$i > i^{(m)} > \delta > d^{(m)} > d$$

$\lim_{m\to\infty} \log(1+i)\, e^{\frac{1}{m}\log(1+i)}$

$i^{(\infty)} = \delta = \log(1+i)$

$= \log(1+i)$

100,000 $\qquad t=5$

$100000(1.11)^5$ $\qquad i=0.11$ $\qquad [168{,}505.82]$

$\left(1+\frac{0.10}{12}\right)^{12\cdot 5}$ $\qquad i^{(12)}=0.10$ $\qquad [164{,}530.89]$

$e^{0.04\cdot 5}$ $\qquad \delta=0.04$ $\qquad [122{,}140.28]$

$\left(1-\frac{0.08}{4}\right)^{-5\cdot 4}$ $\qquad d^{(4)}=0.08$ $\qquad [149{,}788.51]$

Force of Interest

Assuming that the interest rate is variable, you may be interested in looking at the interest rate over short periods of time. That interest rate is:

$$i_{[t,t+1/m]} = \frac{a(t+1/m) - a(t)}{a(t)}$$

And the nominal interest rate is

$$\frac{\left(\frac{a(t+1/m)-a(t)}{a(t)}\right)}{1/m} = \frac{\left(\frac{a(t+1/m)-a(t)}{1/m}\right)}{a(t)}$$

Which as $m \to \infty$ tends to

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{d}{dt}\log a(t)$$

$\frac{d}{dt} \log(f(t))$

$\frac{1}{f(t)} f'(t)$

Force of Interest Examples

Simple interest: $a(t) = 1 + rt \qquad \delta_t = \dfrac{r}{1 + rt}$

Compound interest: $a(t) = (1 + i)^t \qquad \delta_t = \log(1 + i)$

Using Force of Interest

When using a dynamic force of interest:

$$a(t) = \exp\left\{\int_0^t \delta_t dt\right\}$$

If $\delta_t = \delta$ then:

$$a(t) = \exp\left\{\int_0^t \delta dt\right\} = e^{t\delta}$$

Compound Interest:

$$\delta_t = \log(1+i) \rightarrow a(t) = \exp\left\{\int_0^t \log(1+i) dt\right\} = e^{t \log(1+i)} = (1+i)^t$$

$\delta_t = 0.001 + 0.01t + 0.002t^2$

1000 at time 0

Bal at time 4

$$a(t) = \exp\left\{\int_0^t \delta_t dt\right\}$$

$$e^{\int_0^t \delta_t dt}$$

$$1000 \cdot e^{\int_0^4 0.001 + 0.01t + 0.002t^2 dt}$$

$$= 1000\, e^{\left[0.001t + \frac{0.01t^2}{2} + \frac{0.002t^3}{3}\right]_0^4}$$

$$= 1000\, e^{\left[0.001(4) + \frac{0.01(16)}{2} + \frac{0.002(64)}{3}\right]}$$

$$= 1{,}135.04$$